Chapter 6 Appendix: The Calculus of Cost Minimization

In the appendix to Chapter 4, we saw that calculus makes solving the consumer's optimization problem easier. The benefits of using calculus also extend to the firm's constrained optimization problem, cost minimization. Cost minimization is analogous to expenditure minimization for the consumer, and the exercise of solving the producer's optimization problem will serve as a refresher in constrained optimization problems.

Marginal Products of Inputs and Marginal Rate of Technical Substitution

To begin solving the firm's cost-minimization problem, we start with the Cobb–Douglas functional form (as we did in the Chapter 4 appendix). In this case, we use the production function that relates output (Q) to the amount of inputs capital (K) and labor (L): $Q = AK^{\alpha}L^{1-\alpha}$, where $0 < \alpha < 1$, and where the parameter for total factor productivity, A, is greater than zero. We have been relying almost exclusively on the Cobb–Douglas functional form throughout our calculus discussions because this functional form corresponds closely with the assumptions we make about the consumer and the producer. In the context of the producer, the Cobb–Douglas production function satisfies all the assumptions we've made about capital, labor, and firm output, while still yielding simple formulas. In addition, we have chosen a Cobb–Douglas function with another unique property: Because the exponents on K and L (α, $1 - \alpha$) sum to 1, the production function exhibits constant returns to scale.

Before we jump into the producer's cost-minimization problem, let's confirm that the Cobb–Douglas production function satisfies the assumptions about the marginal products of labor and capital and the marginal rate of technical substitution (MRTS). Specifically, we need to show first that the marginal products of labor and capital are positive, and that they exhibit diminishing marginal returns. Next, we will confirm that the MRTS is the ratio of the two marginal products.

Consider first the concept of the marginal product of capital, or how much extra output is produced by using an additional unit of capital. Mathematically, the marginal product of capital is the partial derivative of the production function with respect to capital. It's a partial derivative because we are holding the amount of labor constant. The marginal product of capital is

$$MP_K = \frac{\partial Q(K,L)}{\partial K} = \frac{\partial (AK^{\alpha}L^{1-\alpha})}{\partial K} = \alpha AK^{\alpha-1}L^{1-\alpha}$$

Similarly, the marginal product of labor is

$$MP_L = \frac{\partial Q(K,L)}{\partial L} = \frac{\partial (AK^{\alpha}L^{1-\alpha})}{\partial L} = (1-\alpha)AK^{\alpha}L^{-\alpha}$$

Note that the marginal products above are positive whenever both capital and labor are greater than zero (any time output is greater than zero). In other words, the MP_L and MP_K of the Cobb–Douglas production function satisfy an important condition of production—that output increases as the firm uses more inputs.

We also need to show that the assumptions about the diminishing marginal returns of capital and labor hold true; that is, the marginal products of capital and labor decrease as the amounts of those inputs increase, respectively, holding all else equal. To see this, take the second partial derivative of the production function with respect to each input. In other words, we are taking a partial derivative of each of the marginal products with respect to its input:

$$\frac{\partial^2 Q(K,L)}{\partial K^2} = \frac{\partial MP_K}{\partial K} = \frac{\partial(\alpha AK^{\alpha-1}L^{1-\alpha})}{\partial K} = \alpha(\alpha-1)AK^{\alpha-2}L^{1-\alpha} = -\alpha(1-\alpha)AK^{\alpha-2}L^{1-\alpha}$$

$$\frac{\partial^2 Q(K,L)}{\partial L^2} = \frac{\partial MP_L}{\partial L} = \frac{\partial[(1-\alpha)AK^{\alpha}L^{-\alpha}]}{\partial L} = -\alpha(1-\alpha)AK^{\alpha}L^{-\alpha-1}$$

As long as K and L are both greater than zero (i.e., as long as the firm is producing output), both of these second derivatives are negative, so the marginal product of each input decreases as the firm uses more of the input. Thus, the Cobb–Douglas production function meets our assumptions about diminishing marginal returns to both labor and capital.

We also know from the chapter that the marginal rate of technical substitution and the marginal products of capital and labor are interrelated. In particular, the MRTS shows the change in labor necessary to keep output constant if the quantity of capital changes (or the change in capital necessary to keep output constant if the quantity of labor changes). The MRTS equals the ratio of the two marginal products. To show this is true using calculus, first recognize that each isoquant represents some fixed level of output, say, $\overline{Q}$, so that $Q = Q(K, L) = \overline{Q}$. Begin by totally differentiating the production function:

$$dQ = \frac{\partial Q(K,L)}{\partial K}dK + \frac{\partial Q(K,L)}{\partial L}dL$$

We know that dQ equals zero because the quantity is fixed at $\overline{Q}$:

$$dQ = \frac{\partial Q(K,L)}{\partial K}dK + \frac{\partial Q(K,L)}{\partial L}dL = 0$$

so that

$$\frac{\partial Q(K,L)}{\partial K}dK = -\frac{\partial Q(K,L)}{\partial L}dL$$

Now rearrange to get $-\frac{dK}{dL}$ on one side of the equation:

$$-\frac{dK}{dL} = \frac{\frac{\partial Q(K,L)}{\partial L}}{\frac{\partial Q(K,L)}{\partial K}} = \frac{MP_L}{MP_K}$$

The left-hand side of this equation is the negative of the slope of the isoquant, or the marginal rate of technical substitution.[6] Therefore,

$$MRTS_{LK} = \frac{MP_L}{MP_K}$$

In particular, we differentiate the Cobb–Douglas production function, $Q = AK^{\alpha}L^{1-\alpha}$, and set dQ equal to zero:

$$dQ = \frac{\partial Q(K,L)}{\partial K}dK + \frac{\partial Q(K,L)}{\partial L}dL = \alpha AK^{\alpha-1}L^{1-\alpha}dK + (1-\alpha)AK^{\alpha}L^{-\alpha}dL = 0$$

[6] Recall that isoquants have negative slopes; therefore, the negative of the slope of the isoquant, the $MRTS$, is positive.

Again, rearrange to get $-\frac{dK}{dL}$ on one side of the equation:

$$MRTS_{LK} = -\frac{dK}{dL} = \frac{(1-\alpha)AK^{\alpha}L^{-\alpha}}{\alpha AK^{\alpha-1}L^{1-\alpha}} = \frac{MP_L}{MP_K}$$

which simplifies to

$$MRTS_{LK} = \frac{(1-\alpha)}{\alpha}\frac{K}{L}$$

Thus, we can see that the marginal rate of technical substitution equals the ratios of the marginal products for the Cobb–Douglas production function. This also shows that the $MRTS_{LK}$ decreases as the firm uses more labor and less capital, holding output constant, as we learned in the chapter. Using calculus makes it clear, however, that the rate at which labor and capital can be substituted is determined by α, the relative productivity of capital.

Cost Minimization Using Calculus

Now that we have verified the usefulness of the Cobb–Douglas function for modeling production, let's turn to the firm's cost-minimization problem. Once again, we are faced with a constrained optimization problem: The objective function is the cost of production, and the constraint is the level of output. The firm's goal is to spend the least amount of money to produce a specific amount of output. This is the producer's version of the consumer's expenditure-minimization problem.

As we saw with the consumer's problem, there are two approaches to solving the cost-minimization problem. The first is to apply the cost-minimization condition that we derived in the chapter. At the optimum, the marginal rate of technical substitution equals the ratio of the input prices, wages (W) and capital rental rate (R). We just showed that the marginal rate of technical substitution is the ratio of the marginal products, so the cost-minimization condition is

$$MRTS_{LK} = \frac{MP_L}{MP_K} = \frac{W}{R}$$

For our Cobb–Douglas production function above, finding the optimum solution is easy using this relationship between the marginal rate of technical substitution and the input prices. We start by solving for K as a function of L using the equation for the marginal rate of technical substitution above:

$$\frac{MP_L}{MP_K} = \frac{(1-\alpha)}{\alpha}\frac{K}{L} = \frac{W}{R}$$

$$K = \left[\frac{\alpha}{(1-\alpha)}\frac{W}{R}\right]L$$

Next, plug K into the production constraint to solve for the optimum quantity of labor L^*:

$$\bar{Q} = \alpha AK^{\alpha}L^{1-\alpha} = A\left[\frac{\alpha}{(1-\alpha)}\frac{W}{R}L\right]^{\alpha}L^{1-\alpha}$$

$$\bar{Q} = A\left[\frac{\alpha}{(1-\alpha)}\frac{W}{R}\right]^{\alpha}L^{\alpha}L^{1-\alpha}$$

$$L^* = \left[\frac{(1-\alpha)}{\alpha}\frac{R}{W}\right]^{\alpha}\frac{\bar{Q}}{A}$$

Now solve for K^* by plugging L^* into the earlier expression for K as a function of L:

$$K^* = \left[\frac{\alpha}{(1-\alpha)}\frac{W}{R}\right]L^*$$

$$= \left[\frac{\alpha}{(1-\alpha)}\frac{W}{R}\right]\left[\frac{(1-\alpha)}{\alpha}\frac{R}{W}\right]^{\alpha}\frac{\bar{Q}}{A}$$

We can simplify this expression by inverting the term in the second set of brackets:

$$K^* = \left[\frac{\alpha}{(1-\alpha)}\frac{W}{R}\right]\left[\frac{\alpha}{(1-\alpha)}\frac{W}{R}\right]^{-\alpha}\frac{\bar{Q}}{A}$$

and combining the first and second terms:

$$K^* = \left[\frac{\alpha}{(1-\alpha)}\frac{W}{R}\right]^{1-\alpha}\frac{\bar{Q}}{A}$$

Thus, we have found that the cheapest way of producing $\bar{Q}$ units of output is to use $\left[\frac{\alpha}{(1-\alpha)}\frac{W}{R}\right]^{1-\alpha}\frac{\bar{Q}}{A}$ units of capital and $\left[\frac{(1-\alpha)}{\alpha}\frac{R}{W}\right]^{\alpha}\frac{\bar{Q}}{A}$ units of labor.

Now let's use a second approach to solve for the cost-minimizing bundle of capital and labor: the constrained optimization problem. In particular, the firm's objective, as before, is to minimize costs subject to its production function:

$$\min_{K,L} C = RK + WL \text{ s.t. } \bar{Q} = AK^{\alpha}L^{1-\alpha}$$

Next, write this constrained optimization problem as a Lagrangian so that we can solve for the first-order conditions:

$$\min_{K,L,\lambda} \mathcal{L}(K,L,\lambda) = RK + WL + \lambda(\bar{Q} - AK^{\alpha}L^{1-\alpha})$$

Now take the first-order conditions of the Lagrangian:

$$\frac{\partial\mathcal{L}}{\partial K} = R - \lambda(\alpha AK^{\alpha-1}L^{1-\alpha}) = 0$$

$$\frac{\partial\mathcal{L}}{\partial L} = W - \lambda[(1-\alpha)AK^{\alpha}L^{-\alpha}] = 0$$

$$\frac{\partial\mathcal{L}}{\partial \lambda} = \bar{Q} - AK^{\alpha}L^{1-\alpha} = 0$$

Notice that λ is in both of the first two conditions. Let's rearrange to solve for λ:

$$R = \lambda(\alpha AK^{\alpha-1}L^{1-\alpha})$$

$$\lambda = \frac{R}{\alpha AK^{\alpha-1}L^{1-\alpha}}$$

$$W = \lambda\left[(1-\alpha)AK^{\alpha}L^{-\alpha}\right]$$

$$\lambda = \frac{W}{(1-\alpha)AK^{\alpha}L^{-\alpha}}$$

Now set these two expressions for λ equal to one another:

$$\lambda = \frac{R}{\alpha AK^{\alpha-1}L^{1-\alpha}} = \frac{W}{(1-\alpha)AK^{\alpha}L^{-\alpha}}$$

How can we interpret λ in the context of the firm's cost-minimization problem? In general, the Lagrange multiplier is the value of relaxing the constraint by 1 unit. Here, the constraint is the quantity of output produced; if you increase the given output quantity by 1 unit, the total cost of production at the optimum increases by λ dollars. In other words, λ has a very particular economic interpretation: It is the marginal cost of production, or the extra cost of producing an additional unit of output when the firm is minimizing its costs. We can see that in our λ's above: It is the cost of an additional unit of capital (or labor) divided by the additional output produced by that unit. In Chapter 7, we develop other ways to find marginal costs, but it's good to keep in mind that marginal cost *always* reflects the firm's cost-minimizing behavior.

We can get another perspective on cost minimization by inverting the expressions for λ:

$$\frac{\alpha A K^{\alpha-1} L^{1-\alpha}}{R} = \frac{(1-\alpha) A K^{\alpha} L^{-\alpha}}{W}$$

This relationship shows us precisely what we know is true at the optimum, that $\frac{MP_K}{R} = \frac{MP_L}{W}$, which we can rearrange to get the cost-minimization condition:

$$\frac{W}{R} = \frac{MP_L}{MP_K} = MRTS_{LK}$$

To solve for the optimal bundle of inputs that minimizes cost, we can first solve for K as a function of L:

$$\frac{K^{\alpha}}{K^{\alpha-1}} = \frac{W(\alpha L^{1-\alpha})}{(1-\alpha) R L^{-\alpha}}$$

$$K = \left[\frac{\alpha}{(1-\alpha)}\frac{W}{R}\right] L$$

Plug K as a function of L into the third first-order condition, the constraint:

$$\overline{Q} - A K^{\alpha} L^{1-\alpha} = \overline{Q} - A\left[\frac{\alpha}{(1-\alpha)}\frac{W}{R} L\right]^{\alpha} L^{1-\alpha} = 0$$

Now solve for the cost-minimizing quantity of labor, L^*:

$$A\left[\frac{\alpha}{(1-\alpha)}\frac{W}{R}\right]^{\alpha} L^{\alpha} L^{1-\alpha} = A\left[\frac{\alpha}{(1-\alpha)}\frac{W}{R}\right]^{\alpha} L = \overline{Q}$$

$$L^* = \left[\frac{\alpha}{(1-\alpha)}\frac{W}{R}\right]^{-\alpha} \frac{\overline{Q}}{A} = \left[\frac{(1-\alpha)}{\alpha}\frac{R}{W}\right]^{\alpha} \frac{\overline{Q}}{A}$$

Substitute L^* into our expression for K as a function of L:

$$K^* = \left[\frac{\alpha}{(1-\alpha)}\frac{W}{R}\right] L^* = \left[\frac{\alpha}{(1-\alpha)}\frac{W}{R}\right]\left[\frac{(1-\alpha)}{\alpha}\frac{R}{W}\right]^{\alpha} \frac{\overline{Q}}{A}$$

To simplify, invert the second term and combine:

$$K^* = \left[\frac{\alpha}{(1-\alpha)}\frac{W}{R}\right]\left[\frac{\alpha}{(1-\alpha)}\frac{W}{R}\right]^{-\alpha} \frac{\overline{Q}}{A} = \left[\frac{\alpha}{(1-\alpha)}\frac{W}{R}\right]^{1-\alpha} \frac{\overline{Q}}{A}$$

So using the Lagrangian, we arrive at the same optimal levels of labor and capital that we found using the cost-minimization condition:

$$L^* = \left[\frac{(1-\alpha)}{\alpha}\frac{R}{W}\right]^{\alpha} \frac{\overline{Q}}{A}$$

$$K^* = \left[\frac{\alpha}{(1-\alpha)}\frac{W}{R}\right]^{1-\alpha} \frac{\overline{Q}}{A}$$

figure it out 6A.1

A firm has the production function $Q = 20K^{0.2}L^{0.8}$, where Q measures output, K represents machine hours, and L measures labor hours. If the rental rate of capital is $R = \$15$, the wage rate is $W = \$10$, and the firm wants to produce 40,000 units of output, what is the cost-minimizing bundle of capital and labor?

Solution:

We could solve this problem using the cost-minimization condition. But let's solve it using the Lagrangian, so we can get more familiar with that process. First, we set up the firm's cost-minimization problem as

$$\min_{K,L} C = 15K + 10L \text{ s.t. } 40{,}000 = 20K^{0.2}L^{0.8} \text{ or}$$

$$\min_{K,L,\lambda} \mathcal{L}(K,L,\lambda) = 15K + 10L + \lambda(40{,}000 - 20K^{0.2}L^{0.8})$$

Find the first-order conditions for the Lagrangian:

$$\frac{\partial \mathcal{L}}{\partial K} = 15 - \lambda(4K^{-0.8}L^{0.8}) = 0$$

$$\frac{\partial \mathcal{L}}{\partial L} = 10 - \lambda(16K^{0.2}L^{-0.2}) = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 40{,}000 - 20K^{0.2}L^{0.8} = 0$$

Solve for L as a function of K using the first two conditions:

$$\lambda = \frac{15}{4K^{-0.8}L^{0.8}} = \frac{10}{16K^{0.2}L^{-0.2}}$$

$$15(16K^{0.2}L^{-0.2}) = 10(4K^{-0.8}L^{0.8})$$

$$240(K^{0.2}K^{0.8}) = 40(L^{0.2}L^{0.8})$$

$$L = 6K$$

Now plug L into the third first-order condition and solve for the optimal number of labor and machine hours, L^* and K^*:

$$40{,}000 - 20K^{0.2}L^{0.8} = 0$$

$$20K^{0.2}(6K)^{0.8} = 40{,}000$$

$$20(6)^{0.8}K = 40{,}000$$

$$K^* \approx 477 \text{ machine hours}$$

$$L^* \approx 6(477) \approx 2{,}862 \text{ labor hours}$$

At the optimum then, the firm will use approximately 477 machine hours and 2,862 labor hours to produce 40,000 units. But once again, the Lagrangian provides us with one additional piece of information: the value of λ, or marginal cost:

$$\lambda = \frac{15}{4K^{-0.8}L^{0.8}} = \frac{15}{4(477^{-0.8})(2{,}862^{0.8})} \approx \$0.89$$

Therefore, if the firm wants to produce just one more unit of output—its 40,001st unit of output, to be precise—it would have to spend an additional \$0.89.

The Firm's Expansion Path

So far, we have only solved the firm's cost-minimization problem for a specific quantity. In other words, we've assumed that the firm knows how much output it wants to produce and then decides how best to produce that quantity at the lowest cost. But it might make sense to expand our thinking about how the firm makes its production decisions. In particular, what if a firm wants to know how its optimal input mix varies with its output quantity? This is the firm's expansion path, and it's something we found graphically in the chapter. Recall that an expansion path shows the cost-minimizing relationship between K and L for all possible levels of output. Let's now find the expansion path using calculus.

Consider again the firm with the familiar Cobb–Douglas production function, $Q = AK^{\alpha}L^{1-\alpha}$, and rental cost of capital and wage equal to R and W, respectively. First, write out the constrained optimization problem and the Lagrangian. Note that, unlike before, we are not going to assume that Q is a fixed level of output. In the expansion path, quantity is a variable, and that is reflected in the way we set up the constrained optimization problem below:

$$\min_{K,L} C = RK + WL \text{ s.t. } Q = AK^{\alpha}L^{1-\alpha}$$

$$\min_{K,L,\lambda} \mathcal{L}(K,L,\lambda) = RK + WL + \lambda(Q - AK^{\alpha}L^{1-\alpha})$$

Take the first-order conditions for the Lagrangian:

$$\frac{\partial \mathcal{L}}{\partial K} = R - \lambda(\alpha AK^{\alpha-1}L^{1-\alpha}) = 0$$

$$\frac{\partial \mathcal{L}}{\partial L} = W - \lambda[(1-\alpha)AK^{\alpha}L^{-\alpha}] = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = Q - AK^{\alpha}L^{1-\alpha} = 0$$

As we saw earlier, solving the first two conditions gives us the optimal value of capital K^* as a function of L^*:

$$K^* = \left[\frac{\alpha}{(1-\alpha)}\frac{W}{R}\right]L^*$$

What does this tell us? Given a set of input prices, we now know the cost-minimizing amount of capital at every quantity of labor. The combination of labor and capital then determines the quantity of output. So, what have we found? The expansion path! We could also solve for the optimal amount of labor for every quantity of capital, but it's easier to graph the expansion path with K^* as a function of L^*. Notice that any Cobb–Douglas production function with exponents α and $(1-\alpha)$ generates a linear expansion path with slope

$$\frac{\alpha}{(1-\alpha)}\frac{W}{R}$$

This linear expansion path is yet *another* useful property of the Cobb–Douglas functional form.

figure it out 6A.2

Using the information from Figure It Out 6A.1, derive the firm's expansion path.

Solution:

Because we've already solved the expansion path for the generalized Cobb–Douglas production function, we can plug in the parameters from the firm's cost-minimization problem ($\alpha = 0.2$, $W = \$10$, $R = \$15$) into the equation for the expansion path we found above:

$$K^* = \left[\frac{\alpha}{(1-\alpha)}\frac{W}{R}\right]L^* = \frac{0.2(10)}{0.8(15)}L^* = 0.167L^*$$

Therefore, when minimizing costs, this firm will always choose a combination of inputs in which there is 6 times as much labor as capital, no matter what its desired output is.

Problems

1. For the following production functions,
   - Find the marginal product of each input.
   - Determine whether the production function exhibits diminishing marginal returns to each input.
   - Find the marginal rate of technical substitution and discuss how $MRTS_{LK}$ changes as the firm uses more L, holding output constant.

   a. $Q(K, L) = 3K + 2L$
   b. $Q(K, L) = 10K^{0.5}L^{0.5}$
   c. $Q(K, L) = K^{0.25}L^{0.5}$

2. A more general form of the Cobb–Douglas production function is given by

   $$Q = AK^{\alpha}L^{\beta}$$

   where A, α, and β are positive constants.

   a. Solve for the marginal products of capital and labor.
   b. For what values of α and β will the production function exhibit diminishing marginal returns to capital and labor?
   c. Solve for the marginal rate of technical substitution.

3. Catalina Films produces video shorts using digital editing equipment (K) and editors (L). The firm has the production function $Q = 30K^{0.67}L^{0.33}$, where Q is the hours of edited footage. The wage is \$25, and the rental rate of capital is \$50. The firm wants to produce 3,000 units of output at the lowest possible cost.

   a. Write out the firm's constrained optimization problem.
   b. Write the cost-minimization problem as a Lagrangian.
   c. Use the Lagrangian to find the cost-minimizing quantities of capital and labor used to produce 3,000 units of output.
   d. What is the total cost of producing 3,000 units?
   e. How will total cost change if the firm produces an additional unit of output?

4. A firm has the production function $Q = K^{0.4}L^{0.6}$. The wage is \$60, and the rental rate of capital is \$20. Find the firm's long-run expansion path.